

On-shell symmetries

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Abstract: We define on-shell symmetries and characterize them for Lagrangian systems. The terms appearing in the variation of the Poincaré-Cartan form, which vanish because of field equations, are found to be strongly constrained if the space of solutions has to be preserved. The behaviour with respect to solution dragging is also investigated in order to discuss relations with the theory of internal symmetries of a PDE.

1. Introduction

According to the most general definition, a symmetry of a differential equation is a transformation which preserves the space of solutions. If the equation is variational then symmetries (or, more precisely, suitable specific subsets of all symmetries, e.g. Lagrangian symmetries) can be more conveniently discussed in terms of finite dimensional spaces called *jet prolongations*. A satisfactory geometrical framework for Lagrangian symmetries is well established for all (possibly higher order) field theories (see [1] and references quoted therein).

However, ordinary Lagrangian symmetries are considerably less general than generic symmetries. First of all they are induced by projectable vector fields on the configuration bundle; moreover they are usually required to leave the Lagrangian (or some Lepagean equivalent object) invariant. On the contrary, a generic symmetry is a transformation on the solution space and it is easy to see that there are a number of such transformations which are not induced by vector fields on the configuration bundle; moreover, one can easily work out symmetries which preserve field equations without preserving the Lagrangian (e.g. preserving the Lagrangian modulo pure divergences which, as is well-known, do not influence field equations). As shown in [1], a reasonable generality in defining *generalized Lagrangian symmetries* can be achieved by allowing higher order vector fields which preserve the Poincaré-Cartan form modulo contact forms and exact forms.

Nevertheless, other interesting examples can be found outside this last framework. To see this, let us consider –as a pedagogical example– the free particle in one dimension,

described by the Lagrangian

$$L_{fp} = \frac{1}{2}v^2 \quad (1.1)$$

over the bundle $\mathbb{R} \times T\mathbb{R}$ endowed with fibered coordinates (t, q, v) . For later convenience we shall also consider higher order tangent bundles, e.g. $\mathbb{R} \times T^2\mathbb{R}$ endowed with fibered coordinates (t, q, v, a) , $\mathbb{R} \times T^3\mathbb{R}$ endowed with fibered coordinates (t, q, v, a, b) and so on. Let us consider the following infinitesimal transformation $\delta q = \lambda v + q$ ($\lambda \neq 0$), which is naturally prolonged to all orders as follows:

$$\begin{cases} \delta q = \lambda v + q \\ \delta v = \lambda a + v \\ \delta a = \lambda b + a \\ \dots \end{cases} \quad (1.2)$$

This transformation preserves the solutions of the free particle Euler-Lagrange equation without preserving the equation of motion itself. In fact, if one deforms the equation of motion $a = 0$ along the transformation (1.2) the result $\delta a = \lambda b + a$ is identically vanishing along solutions. In [1] it was shown how to regard this infinitesimal transformation as a higher order vector field

$$\Xi = (\lambda v + q) \frac{\partial}{\partial q} + (\lambda a + v) \frac{\partial}{\partial v} + (\lambda b + a) \frac{\partial}{\partial a} + \dots \quad (1.3)$$

If we consider the deformation of the Lagrangian along the transformation (1.2) we obtain

$$\delta L_{fp} = v\delta v = v(\lambda a + v) = v^2 + \lambda va = -aq + \frac{d}{dt} \left(vq + \frac{\lambda}{2}v^2 \right) \quad (1.4)$$

We stress that there exists an alternative and inequivalent splitting of the Lagrangian variation δL_{fp} into a total derivative and a term vanishing on-shell. It is simply given by the *first variation formula* (see, e.g., [1]), which in this case reads:

$$\delta L_{fp} = v\delta v = -a\delta q + \frac{d}{dt} (v\delta q) = -a(\lambda v + q) + \frac{d}{dt} (\lambda v^2 + qv) \quad (1.5)$$

This second splitting can be called the *trivial splitting*, since such a splitting exists in fact for all Lagrangians and all transformations. Going back to our simple example, notice now that we have been able to show that the Lagrangian remains invariant modulo a total derivative and a term vanishing on-shell in (at least) two different and inequivalent ways.

As was already well-known in the literature (see [2], [3], [4]) any one of these splittings is enough to implement Nöther theorem. In fact, if the Lagrangian is invariant in the weak sense of

$$\delta L = \alpha^i E_i + \frac{df}{dt} \quad (1.6)$$

then identity (1.6) can be easily recasted as a conservation law

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial u^i} \delta q^i - f \right) = (\alpha^i - \delta q^i) E_i \quad (1.7)$$

Reverting then to the free particle, the two splittings above produce by Nöther theorem the following first integrals of motion, respectively:

$$\mathcal{E}_{(1.4)} = v(\lambda v + q) - vq - \frac{\lambda}{2}v^2 = \frac{\lambda}{2}v^2 \quad \mathcal{E}_{(1.5)} \equiv 0 \quad (1.8)$$

The first one being basically the energy of the particle; the second one being trivially “*conserved*” (along any, possibly non-critical, curve) since it is a constant. Because of the fact that the trivial splitting actually produces a conservation law of a very trivial character, it is clear why little effort has been devoted to characterize trivial splittings together with their behaviour with respect to solution dragging.

Other more general examples of such a behaviour can be obtained by considering the class of transformations

$$\delta q = B(v) + A(v)q \quad (1.9)$$

where $A(v)$ and $B(v)$ are arbitrary functions. We also remark that there exist transformations which also allow a non-trivial splitting as above, although they do not preserve the solution space. As an example, check the following:

$$\delta q = \lambda v + q^2 \quad (1.10)$$

Passing from Mechanics to Field Theory more physically relevant examples can be found. For instance, supersymmetries in the Rarita-Schwinger model (as well as in other supergravity models; see [5]) are known to be special kinds of *symmetry transformations*, usually called *on-shell symmetries*. Remarkably enough, the definition which seems to be implicitly assumed for this notion is the following: *a transformation leaving the Lagrangian invariant on-shell modulo pure divergences*.

However, a simple argument shows that this naive attitude is indeed untenable. First of all it has to be remarked that *all transformations* leave *any* Lagrangian invariant modulo pure divergences and on-shell terms, just because of the first variation formula (see equation (3.2) below).

Of course, when a transformation is claimed in literature to be a symmetry on-shell some splitting is usually exhibited for the variation of the Lagrangian; however, its non-triviality is hardly ever proven, while, as we see from the above example, it is clear that a *non-trivial* splitting is strictly speaking necessary. Nevertheless, we shall see that a non-trivial splitting is not at all *sufficient*, since further requirements are needed for a transformation to be a symmetry on-shell. We shall in fact show that in some cases, even when a non-trivial splitting is exhibited, the transformation might not preserve solutions.

The present paper is therefore devoted to characterize non-trivial on-shell symmetries in Field Theory (in particular in Mechanics) and to provide a geometrical picture able to encompass higher order vector fields as infinitesimal transformations of some kind (we remark that the higher order vector field (1.3) does not allow a flow on any $\mathbb{R} \times T^k \mathbb{R}$ nor on the inverse limit $\mathbb{R} \times T^\infty \mathbb{R}$; see below). The main result of the present paper is contained in Definition (3.17) which is, to our knowledge, new in the physically oriented literature on the subject.

As a technical tool we shall use jet bundles and Poincaré-Cartan forms (see [1], [6], [7], [8]). We stress that these are suitable mathematical tools though, in a sense, unessential to our analysis. Our results can be in fact easily translated back into the usual language of Lagrangian functionals, though loosing some of the geometrical understanding and making some steps considerably more cumbersome.

2. Notation

We assume the reader is already familiar with bundle language. Standard references can be found in [8], [9] and references quoted therein. A field theory is defined on a configuration bundle (B, M, π, F) with local coordinates (x^μ, y^i) , $\mu = 1 \dots m = \dim(M)$ and $i = 1 \dots n = \dim(F)$. Configurations are sections $\sigma : M \rightarrow B$ ($\pi \circ \sigma = \text{id}_M$). The bundle of vertical vectors is denoted by $V(\pi)$, π being the projection of the relevant bundle. The Lie derivative of a section σ with respect to a (higher order) vector field Ξ projecting onto an ordinary vector field ξ is defined as

$$\mathcal{L}_\Xi \sigma = T\sigma(\xi) - \Xi \circ \sigma \equiv (\mathcal{L}_\Xi y^i) \partial_i \quad \partial_i = \frac{\partial}{\partial y^i} \quad (2.1)$$

The jet prolongations $J^k B$ take the derivative of fields up to order k into account. We denote by π_h^k the projection of $J^k B$ onto $J^h B$ ($k > h$), by π_0^k the projection over B and by $\pi^k = \pi \circ \pi_0^k$ the projection over M . Fibered coordinates on $J^k B$ will be denoted by $(x^\mu, y^i, y_\mu^i, y_{\mu\nu}^i, \dots, y_{\mu_1 \dots \mu_k}^i)$, with obvious symmetries in their lower indices.

Being J^k a functor, a strong bundle morphism of B (i.e. a fibered morphism projecting onto a diffeomorphism of M) can be canonically prolonged to a bundle morphism of $J^k B$. As a consequence vector fields and sections of B can be canonically prolonged as well to $J^k B$. Sections of $J^k B$ which are the prolongation of some section σ of B are called *holonomic* and they are denoted by $j^k \sigma$.

Forms on $J^k B$ which vanish along holonomic sections are called *contact forms*. Contact 1-forms are generated by

$$\omega^i = dy^i - y_\mu^i dx^\mu, \quad \omega_\lambda^i = dy_\lambda^i - y_{\lambda\mu}^i dx^\mu, \dots \quad (2.2)$$

Contact forms fill a graded ideal in the exterior algebra, where the degree counts for the number of contact 1-forms factors (2.2). For example, $\alpha_{ij} \omega^i \wedge \omega_\mu^j \wedge dx^\mu$ is a 3-form of contact order 2.

A *horizontal form* on $J^k B$ is a form which vanishes when contracted along a vertical vector field. Horizontal forms contain only the differentials of the base coordinates dx^μ . The coordinate basis of horizontal m -forms is denoted by \mathbf{ds} ($m = \dim(M)$); locally $\mathbf{ds} = dx^1 \wedge dx^2 \wedge \dots \wedge dx^m$. By contractions along coordinate vector fields the bases of horizontal $(m-p)$ -forms (with $1 \leq p \leq m$) are generated recursively:

$$\mathbf{ds}_\mu = \partial_\mu \lrcorner \mathbf{ds}, \quad \mathbf{ds}_{\mu\nu} = \partial_\nu \lrcorner \mathbf{ds}_\mu, \quad \dots \quad (2.3)$$

where \lrcorner denotes the interior product between vectors and forms. By pull-back onto $J^{k+1} B$ any form on $J^k B$ can be canonically split into a horizontal and a contact form on $J^{k+1} B$.

The projectors along horizontal and contact forms are denoted by $H(\cdot)$ and $K(\cdot)$, respectively.

The jet bundles $J^k B$ form an inverse family and the inverse limit is an infinite dimensional bundle called the *infinite jet prolongation bundle*; it is denoted by $J^\infty B$. Accordingly, the projection of $J^\infty B$ over $J^k B$ will be denoted by π_k^∞ . This infinite jet bundle is meant to be endowed with the inverse topology and manifold structure.

A *higher order vector field* is a section of the bundle $(\pi_0^k)^* TB \rightarrow J^k B$; see [1]. Locally a projectable higher order vector field has the following form:

$$\Xi = \xi^\mu(x^\lambda) \partial_\mu + \xi^i(x^\lambda, y^i, y_\lambda^i, \dots, y_{\lambda_1 \dots \lambda_k}^i) \partial_i \quad (2.4)$$

and the integer k is called the *order* of Ξ . Higher order vector fields can be prolonged as ordinary vector fields (see [1], [8], [10]).

A *Poincaré-Cartan form* (PC form) of order k is a form Θ on $J^{2k-1} B$ obeying the following three axioms:

$$\begin{aligned} PC1 \quad & \forall X, Y \in V(\pi^{2k-1}), X \lrcorner Y \lrcorner \Theta = 0 \\ PC2 \quad & \forall X \in V(\pi_{k-1}^{2k-1}), X \lrcorner \Theta = 0 \\ PC3 \quad & \forall X \in V(\pi_0^{2k-1}), H(X \lrcorner d\Theta) = 0 \end{aligned}$$

We remark that the space of PC forms is a linear subspace of the space of all forms on $J^{2k-1} B$.

As a consequence of the above axioms the horizontal part of a PC form of order k is necessarily of the form:

$$H(\Theta) = \mathcal{L}(x^\mu, y^i, y_\mu^i, \dots, y_{\mu_1 \dots \mu_k}^i) \mathbf{ds} \quad (2.5)$$

and it is called *the Lagrangian (of order k)* induced by Θ ; the contact part of Θ has in general a more complicated structure (see [7], [8] for details). Often we shall confuse Θ with $(\pi_{2k-1}^\infty)^* \Theta$ which is the expression of the PC form regarded as a form on $J^\infty B$.

If $X = X^i \partial_i + X_\mu^i \partial_i^\mu + \dots \in V(\pi^{2k-1})$ is a vertical vector then the quantity

$$H(X \lrcorner d\Theta) = X^i E_i \mathbf{ds} \quad E_i \equiv \partial_i \mathcal{L} - d_\mu \partial_i^\mu \mathcal{L} + \dots \quad (2.6)$$

is directly related to field equations $E_i = 0$. Mechanics can be obtained in the special case $k = 1$ and $M \equiv \mathbb{R}$ understanding the base indices which run on a single value.

From now on we shall explicitly consider the case of a first order field theory ($k = 1$), although our calculation easily extend (with the appropriate modifications) to the higher order case. In particular, a first order PC form has necessarily the following structure (see [1], [7], [8]):

$$\Theta = \mathcal{L}(x^\lambda, y^k, y_\lambda^k) \mathbf{ds} + p_i^\mu \omega^i \wedge \mathbf{ds}_\mu \quad p_i^\mu \equiv \partial_i^\mu \mathcal{L}(x^\lambda, y^k, y_\lambda^k) \quad (2.7)$$

where $\mathcal{L}(x^\lambda, y^k, y_\lambda^k)$ is a first order Lagrangian and p_i^μ are its *canonical momenta*.

3. Characterization of on-shell symmetries

Let a *Nöther splitting* (for a PC form Θ) be a decomposition of the Lie derivative of Θ along a higher order vector field Ξ into an exact form $d\alpha$, a contact form $\hat{\omega}$ and an m -form $f(E)$ vanishing on-shell, i.e.

$$\mathcal{L}_\Xi \Theta = d\alpha + \hat{\omega} + f(E) \quad (3.1)$$

where E is a short-cut for the Euler-Lagrange equations of the Lagrangian induced by Θ . It is known that these equations take the form $H(X \lrcorner d\Theta) = 0$ for arbitrary X . The form $f(E)$ will be assumed to be *horizontal* without any loss of generality, since its contact part can be directly included into $\hat{\omega}$.

We stress that the existence of a Nöther splitting is not a restrictive condition on Ξ , nor on Θ . In fact, one can trivially show that a Nöther splitting always exists for *all* higher order vector fields $\Xi = \xi^\mu \partial_\mu + \xi^i \partial_i$ and for *all* PC forms Θ . In fact, one has

$$\mathcal{L}_\Xi \Theta = d(\Xi \lrcorner \Theta) + K(\Xi \lrcorner d\Theta) + H(\Xi \lrcorner d\Theta) \quad (3.2)$$

which is a Nöther splitting since $H(\Xi \lrcorner d\Theta) = (\mathcal{L}_\Xi y^i) E_i \mathbf{ds}$ does in fact vanish on-shell (being $k = 1$, here we have set $\mathcal{L}_\Xi y^i \equiv \xi^\mu y_\mu^i - \xi^i$).

Nöther splittings are introduced in literature since they express a sufficient condition to obtain conservation laws via Nöther theorem (see, e.g., [2], [3], [4]). We shall investigate whether and when they also preserve solutions.

First of all, let us investigate this matter in a heuristic way. Let us then consider a 1-parameter family of morphisms $\Phi_s : J^r B \rightarrow B$ defined in a neighbourhood of the origin $s \in (-\epsilon, \epsilon)$; furthermore let us assume that for $s = 0$ the morphism Φ_0 reduces to the projection π_0^r . [The heuristic part of the argument is that, when $r = 0$, Φ_s is the flow of an ordinary vector field Ξ . In the event of a higher order vector field the flow (if any exists) would be a flow on the infinite jet prolongation. We shall verify below that the results we are going to prove will keep holding true infinitesimally for all higher order vector fields Ξ , even when they do not allow a flow on $J^\infty B$].

Let us then drag Θ along the flow Φ_s (or, better, along its infinite prolongation $J^\infty \Phi_s$). The form so obtained will be denoted by ${}^s\Theta \equiv (J^\infty \Phi_s)^* \Theta$. We shall here investigate when this flow preserves the solutions (or, better, when it preserves infinitesimally the solutions).

Our program is achieved in three steps:

- (1) first of all we shall canonically associate a 1-parameter family of PC forms ${}^s\Theta_*$ by defining a suitable dragging of Θ along Φ_s within the space of PC forms;
- (2) we shall determine under which conditions a solution of Θ is a solution of all the PC forms ${}^s\Theta_*$ for all $s \in (-\epsilon, \epsilon)$;
- (3) we shall verify that the conditions so determined guarantee that the infinitesimal generator Ξ (which is a higher order vector field) infinitesimally preserves the space of solutions of Θ , even in the event of a higher order vector field which does not allow a flow.

Step 1

The (first order) PC form Θ can be pulled-back along the transformations $J^\infty \Phi_s$ obtaining the forms ${}^s\Theta$. The local expression of a first order transformation is:

$$\begin{cases} x^\lambda = f^\lambda(x') \\ y^k = Y^k(x'^\mu, y'^i, y'_\lambda{}^i) \\ y'_\lambda{}^k = \bar{J}_\lambda^\nu d_\nu Y^k(x'^\mu, y'^i, y'_\lambda{}^i) \\ \dots \end{cases} \quad \begin{cases} J_\lambda^\nu = \partial_\lambda f^\nu \\ J_\lambda^i = \partial_\lambda Y^i \\ J_j^i = \partial_j Y^i \\ J_j^{i\lambda} = \partial_j^\lambda Y^i \end{cases} \quad (3.3)$$

We shall denote by \bar{J}_λ^ν the inverse of J_ν^λ and by J the determinant of J_ν^λ . One can easily obtain:

$${}^s\Theta = J\mathcal{L}d\mathbf{s}' + J\bar{J}_\mu^\nu p_j^\mu J_i^j \omega'^i \wedge d\mathbf{s}'_\nu + J\bar{J}_\mu^\nu p_j^\mu J_i^{j\rho} \omega'^i_\rho \wedge d\mathbf{s}'_\nu \quad (3.4)$$

We shall first investigate when the form (3.4) is a PC form. Axioms *PC1* and *PC2* are trivially satisfied. Axiom *PC3* is not satisfied in general, since for any $X \in V(\pi_0^\infty)$ one has:

$$H(X \lrcorner d {}^s\Theta) = -H(X \lrcorner d(\Delta_i^\mu \omega^i \wedge d\mathbf{s}_\mu)), \quad \Delta_i^\mu \equiv J E_k J_i^{k\mu} \quad (3.5)$$

From (3.5) we see that axiom *PC3* is satisfied by each ${}^s\Theta$ when we restrict to ordinary transformations (i.e., $J_i^{k\mu} = 0$). However, we also see that for a truly higher order transformation there exists a canonical contact form

$$\Delta = \Delta_i^\mu \omega^i \wedge d\mathbf{s}_\mu \quad (3.6)$$

such that ${}^s\Theta_* = {}^s\Theta + \Delta$ is a PC form. In fact, *PC1* and *PC2* hold trivially for each ${}^s\Theta_*$, while *PC3* follows from (3.5). Notice that ${}^s\Theta_*$ and ${}^s\Theta$ differ by a contact form so that they “induce” the same Lagrangian and the same equations.

We shall then define ${}^s\Theta_*$ to be the dragging (within the space of PC forms) of Θ along Φ_s . The fact that all ${}^s\Theta_*$ are PC forms will be important for our purposes, since PC forms induce field equations according to equation (2.6).

Step 2

We have now to implement a condition which ensures the preservation of solutions. The easiest way to do it, though not the most general, consists in requiring ${}^s\Theta_* = \Theta$ for all s ; then the field equations of ${}^s\Theta_*$ are trivially the same field equations of Θ . Notice however that even when ${}^s\Theta_* = \Theta + d\alpha_s$ for some family of $(m-1)$ -forms α_s on $J^\infty B$, still they induce the same field equations. In the Introduction we presented a simple example of a transformation which did not preserve field equations but it still preserved solutions. Generally speaking we have the following Lemma:

Lemma (3.7): let $((a_s)_i^k, (a_s)_i^{k\mu}, (a_s)_i^{k\mu\nu}, \dots)$ be a 1-parameter family of (local) functions on $\mathbb{R} \times J^\infty B$; let the limit for $s \rightarrow 0$ be $(\delta_k^i, 0, 0, \dots)$.

Then σ is a solution for $E_k = 0$ iff $\forall s$, σ is a solution of $(a_s)_i^k E_k + (a_s)_i^{k\mu} d_\mu E_k + (a_s)_i^{k\mu\nu} d_\mu d_\nu E_k + \dots = 0$

Proof: (\Rightarrow) if $E_k = 0$ then $d_\mu E_k = 0, d_{\mu\nu} E_k = 0, \dots$

(\Leftarrow) Set $s = 0$.

According to this Lemma we say that Φ_s preserves solutions iff

$$H(X \lrcorner d {}^s \Theta_*) \equiv X^i E_i^{(s)} \mathbf{ds} = X^i ((a_s)_i^k E_k + (a_s)_i^{k\mu} d_\mu E_k + (a_s)_i^{k\mu\nu} d_\mu d_\nu E_k + \dots) \mathbf{ds} \quad (3.8)$$

A necessary and sufficient condition for preserving solutions is hence

$$X \lrcorner d {}^s \Theta_* = X^i ((a_s)_i^k E_k + (a_s)_i^{k\mu} d_\mu E_k + (a_s)_i^{k\mu\nu} d_\mu d_\nu E_k + \dots) \mathbf{ds} + \omega^{(1)} \quad (3.9)$$

for some m -form $\omega^{(1)}$ of contact order at least 1 and for all $X \in V(\pi \circ \pi_0^\infty)$.

By simply expanding $X \lrcorner \alpha = X^i A_i \mathbf{ds} + \omega^{(1)}$ for a generic $(m+1)$ -form α and $X \in V(\pi \circ \pi_0^\infty)$, equation (3.9) can be recasted as follows:

$$d {}^s \Theta_* = ((a_s)_i^k E_k + (a_s)_i^{k\mu} d_\mu E_k + (a_s)_i^{k\mu\nu} d_\mu d_\nu E_k + \dots) \omega^i \wedge \mathbf{ds} + \omega^{(2)} \quad (3.10)$$

for some form $\omega^{(2)}$ of contact order at least 2.

Recalling now that ${}^s \Theta_* = {}^s \Theta + \Delta$ holds, by taking the infinitesimal generator $d_s|_{s=0}$ and setting $\alpha_i^k = (\dot{a}_s)_i^k$, $\alpha_i^{k\mu} = (\dot{a}_s)_i^{k\mu}$, \dots , equation (3.10) is finally recasted as

$$d\mathcal{L}_\Xi \Theta = (\alpha_i^k E_k + \alpha_i^{k\mu} d_\mu E_k + \alpha_i^{k\mu\nu} d_\mu d_\nu E_k + \dots) \omega^i \wedge \mathbf{ds} - d(E_k \partial_i^\mu \xi^k \omega^i \wedge \mathbf{ds}_\mu) + \omega^{(2)} \quad (3.11)$$

This condition will be called *covariance identity*. We say that Ξ is a symmetry on-shell if the identity (3.11) holds true for some choice of the coefficients α_i^k , $\alpha_i^{k\mu}$, \dots .

We stress that the identity (3.11) is a strict condition on Ξ ; for example, if $d\mathcal{L}_\Xi \Theta$ expands with a term $\alpha_i \omega^i \wedge \mathbf{ds}$ which does not vanish on-shell (as it happens for instance with the free particle and the infinitesimal transformation $\delta q = \lambda v + q^2$ mentioned as an example in the Introduction) then the transformation is not a symmetry.

It is now time to show how condition (3.11) characterizes non-trivial Nöther splittings leading to on-shell symmetries. Let us suppose that $\mathcal{L}_\Xi \Theta = d\alpha + \hat{\omega} + f(E)$. Because of the inverse limit topology fixed on $J^\infty B$, both $\hat{\omega}$ and $f(E)$ are the pull-back of objects on some finite order prolongation. Since we do not need prolongation orders to be sharp, we can assume without any loss of generality that both $\hat{\omega}$ and $f(E)$ truncate at the same order r . Hence

$$\begin{cases} \hat{\omega} = \theta_i^\lambda \omega^i \wedge ds_\lambda + \theta_i^{\lambda\mu} \omega_\mu^i \wedge \mathbf{ds}_\lambda + \dots + \theta_i^{\lambda\mu_1 \dots \mu_r} \omega_{\mu_1 \dots \mu_r}^i \wedge ds_\lambda + \omega^{(2)} \\ f(E) = (c^k E_k + c^{k\mu} d_\mu E_k + \dots + c^{k\mu_1 \dots \mu_r} d_{\mu_1 \dots \mu_r} E_k) \mathbf{ds} \equiv C \mathbf{ds} \end{cases} \quad (3.12)$$

Moreover let us set

$$\alpha_i^k E_k + \alpha_i^{k\mu} d_\mu E_k + \dots + \alpha_i^{k\mu_1 \dots \mu_r} d_{\mu_1 \dots \mu_r} E_k \equiv A_i \quad (3.13)$$

Hence the covariance condition (3.11) can be recasted as

$$d\hat{\omega} + df(E) = A_i \omega^i \wedge \mathbf{ds} - d(E_k \partial_i^\mu \xi^k \omega^i \wedge \mathbf{ds}_\mu) + \omega^{(2)} \quad (3.14)$$

which in turn expands into the following conditions

$$\left\{ \begin{array}{ll} \partial_i C - d_\lambda \theta_i^\lambda = A_i + d_\mu (E_k \partial_i^\mu \xi^k) & \\ \partial_i^\mu C - d_\lambda \theta_i^{\lambda\mu} - \theta_i^\mu = E_k \partial_i^\mu \xi^k & \Rightarrow \theta_i^\mu = \partial_i^\mu C - d_\lambda \partial_i^{\lambda\mu} C - E_k \partial_i^\mu \xi^k \\ \partial_i^{\mu\nu} C - d_\lambda \theta_i^{\lambda\mu\nu} - \theta_i^{\mu\nu} = 0 & \Rightarrow \theta_i^{\mu\nu} = \partial_i^{\mu\nu} C - d_\lambda \partial_i^{\lambda\mu\nu} C \\ \dots & \\ \partial_i^{\mu_2 \dots \mu_r} C - d_\lambda \theta_i^{\lambda \mu_2 \dots \mu_r} - \theta_i^{\mu_2 \dots \mu_r} = 0 & \Rightarrow \theta_i^{\mu_2 \dots \mu_r} = \partial_i^{\mu_2 \dots \mu_r} C - d_\lambda \partial_i^{\lambda \mu_2 \dots \mu_r} C \\ \partial_i^{\mu_1 \dots \mu_r} C - \theta_i^{\mu_1 \dots \mu_r} = 0 & \Rightarrow \theta_i^{\mu_1 \dots \mu_r} = \partial_i^{\mu_1 \dots \mu_r} C \end{array} \right. \quad (3.15)$$

A solution for (3.15) is obtained by substitution, proceeding from the bottom one up to the top one. This shows that the contact form $\hat{\omega}$ cannot be fixed at will but it is uniquely determined by C and the symmetry generator Ξ . The first item of (3.15) can be finally recasted as

$$\mathbb{E}_i(C) = A_i, \quad \mathbb{E}_i \equiv \partial_i - d_\mu \circ \partial_i^\mu + d_{\mu\nu} \circ \partial_i^{\mu\nu} + \dots \quad (3.16)$$

where \mathbb{E}_i is the Euler-Lagrange operator. Hence we stress that $\mathbb{E}_i(C)$ is constrained to vanish on-shell and furthermore to be exactly related to $d\mathcal{L}_\Xi \Theta$.

Hence the following definition is well motivated:

Definition (3.17): a higher order vector field Ξ (of order 1) is a symmetry on-shell of a PC form Θ (of order 1) if and only if there exists a Nöther splitting

$$\mathcal{L}_\Xi \Theta = d\alpha + \hat{\omega} + C \, \mathbf{ds} \quad (3.18)$$

such that C and $\mathbb{E}_i(C)$ both vanish on-shell, $d\mathcal{L}_\Xi \Theta = \mathbb{E}_i(C) \omega^i \wedge \mathbf{ds} - d(E_k \partial_i^\mu \xi^k \omega^i \wedge \mathbf{ds}_\mu) + \omega^{(2)}$ and the contact part $\hat{\omega}$ is uniquely determined as in (3.15).

In particular we stress that providing a Nöther splitting (3.18) and simply checking that C vanishes on-shell (as it is sometimes done in the literature) is definitely insufficient and sometimes drastically wrong.

Step 3

We shall now verify that the definition of on-shell symmetries given above holds for higher order vector fields, without resorting to the (possible) existence of their flows. In particular we shall prove that the variation of field equations along Ξ vanishes along solutions.

Definition (3.19): We say that a (possibly higher order) vector field Ξ does infinitesimally preserve solutions of Θ if for all $X \in V(\pi \circ \pi_0^\infty)$ one has

$$H(\mathcal{L}_\Xi(X \lrcorner d\Theta)) \simeq 0 \quad (3.20)$$

where \simeq means that it does vanish on-shell.

One can easily expand condition (3.20) into the following

$$H(\mathcal{L}_\Xi(X \lrcorner d\Theta)) \simeq -X^i (\partial_k E_i \mathcal{L}_\Xi y^k + \partial_k^\mu E_i \mathcal{L}_\Xi y_\mu^k + \partial_k^{\mu\nu} E_i \mathcal{L}_\Xi y_{\mu\nu}^k) \mathbf{ds} \quad (3.21)$$

modulo terms vanishing on-shell. This quantity can be directly shown to vanish on-shell when Ξ is a symmetry on-shell. In fact, by expanding $d\mathcal{L}_\Xi\Theta$ and requiring the on-shell vanishing of the term along $\omega^i \wedge \mathbf{ds}$ (which is a necessary condition for Ξ to be a symmetry on-shell according to definition (3.17)) one obtains

$$\partial_k E_i \mathcal{L}_\Xi y^k + \partial_k^\mu E_i \mathcal{L}_\Xi y_\mu^k + \partial_k^{\mu\nu} E_i \mathcal{L}_\Xi y_{\mu\nu}^k \simeq 0 \quad (3.22)$$

i.e. along solutions.

We stress that in definition (3.17) as well as in the calculations performed above the flow of Ξ (which might not exist) is never used.

4. General vector fields as flow generators

Higher order vector fields are not vector fields on any finite order prolongation; however, they prolonge to true vector fields on $J^\infty B$. In any event, showing the existence of flows on $J^\infty B$ is not trivial as in the finite dimensional case and there are explicit examples of vector fields on $J^\infty B$ which do not allow a flow, even if restricted to a neighbourhood of the parameter origin $s = 0$.

Despite for Nöther theorem we need infinitesimal transformations only (as we saw in Section 3), nevertheless it could be of interest to see whether there exists a group of transformations associated to the infinitesimal transformation represented by a given higher order vector Ξ .

It can be easily shown that any symmetry on-shell is tangent to the infinite prolongation of field equations $J^\infty E \subset J^\infty B$. In fact, by expanding $\Xi \lrcorner dE_i$ one easily obtains

$$\Xi \lrcorner dE_i = \partial_k E_i \mathcal{L}_\Xi y^k + \partial_k^\mu E_i \mathcal{L}_\Xi y_\mu^k + \partial_k^{\mu\nu} E_i \mathcal{L}_\Xi y_{\mu\nu}^k \quad (4.1)$$

which, as already proven above, does in fact vanish on-shell.

In Mechanics, when the system is non degenerate the prolongation of the equations of motion is a finite dimensional submanifold of $J^\infty B$. Moreover, the equations are normal so that they allow one to express any appearance of derivatives higher than one in terms of Lagrangian coordinates and velocities. As a consequence we obtain a true vector field on the prolongation of the equations, which is a finite dimensional manifold. Accordingly, it follows that on-shell symmetries define in fact a flow on the equations (and in turn they can drag solutions) even when they do not define a flow on the whole prolongation $J^\infty B$.

In Field Theory the situation is a bit more complicated. A vector field is still induced on the submanifold representing the equations, though in general field equations are not normal and the equation submanifold defines an infinite dimensional space. In field theory, therefore, one is not a priori guaranteed that on-shell symmetries do in fact drag configurations nor solutions.

For instance the higher order vector field related to the transformation $\delta q = \lambda v + q$ prolongs and restricts to $J^\infty E$ to obtain

$$\bar{\Xi} = (\lambda v + q) \frac{\partial}{\partial q} + v \frac{\partial}{\partial v} \quad (4.2)$$

which turns out to be a true vector field on $J^\infty E$ (here parametrized by (t, q, v)). This vector field on $J^\infty E$ defines the following flow

$$\begin{cases} t' = t \\ q' = Q(t, q, v) = (q + \lambda v s) e^s \\ v' = V(t, q, v) = v e^s \end{cases} \quad (4.3)$$

This can be directly checked to be a 1-parameter subgroup and to preserve solutions (i.e. uniform linear motions).

On the contrary, the transformation $\delta q = \lambda v + q^2$ restricts to

$$\bar{\Xi} = (\lambda v + q^2) \frac{\partial}{\partial q} + 2vq \frac{\partial}{\partial v} + 2v^2 \frac{\partial}{\partial a} \quad (4.4)$$

which is not tangent to $J^\infty E$.

5. Conclusions and perspectives

We remark that the conditions (3.17) above are quite effective in characterizing the non-trivial on-shell symmetries. For example in the Introduction we presented a number of examples and counterexamples for the free particle. For the transformation (1.2) we have $\mathbb{E}(-qa) = -2a$ which in fact vanishes on-shell. The PC form for the free particle is

$$\Theta = \frac{1}{2}v^2 dt + v \omega \quad (5.1)$$

Hence we obtain

$$\mathcal{L}_{\Xi}\Theta = (v^2 + \lambda va) dt + (\lambda a + 2v) \omega + \lambda v \dot{\omega} = d(vq + \frac{\lambda}{2}v^2) + (\lambda a + v) \omega - q \dot{\omega} - aq dt \quad (5.2)$$

while the differential is

$$d\mathcal{L}_{\Xi}\Theta = (\lambda b + 2a) dt \wedge \omega + \lambda a dt \wedge \dot{\omega} + \omega^{(2)} = -2a \omega \wedge dt + d(\lambda a \omega) + \omega^{(2)} \quad (5.3)$$

Hence, using the notation of Section 2, we have $A = -2a$ and $C = -aq$. Then we can check conditions (3.17); we obtain

$$\mathbb{E}(C) = \mathbb{E}(-aq) = -2a \equiv A \quad (5.4)$$

as required. Furthermore, the contact part ω prescribed by (3.15) is

$$(\lambda a + v) \omega - q \dot{\omega} \quad (5.5)$$

in complete agreement with what we found in (5.2). Hence we can conclude that the transformation (1.2) is in fact a symmetry on-shell.

As in the Introduction we can also consider the transformation $\delta q = \lambda v + q^2$. In this case we obtain

$$\mathcal{L}_{\Xi}\Theta = d(\frac{\lambda}{2}v^2 + vq^2) + (\lambda a + 2qv) \omega - q^2 \dot{\omega} - aq^2 dt \quad (5.6)$$

which we stress is a non-trivial Nöther splitting. Moreover we have

$$d\mathcal{L}_{\Xi}\Theta = d(\lambda a \omega) - (2v^2 + 4qa) \omega \wedge dt + \omega^{(2)} \quad (5.7)$$

In this second case we hence have $C = -aq^2$ and $A = -2v^2 - 4qa$. Accordingly, we have $\mathbb{E}(C) = -4aq - 2v^2 = A$, but A itself does not vanish on-shell as prescribed. We conclude that the transformation $\delta q = \lambda v + q^2$ is not an on-shell symmetry even if a non-trivial Nöther splitting exists.

Further investigation will be devoted to study Rarita-Schwinger supersymmetries.

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